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**LETTER TO THE EDITOR**

# A crossing probability for critical percolation in two dimensions

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**Abstract.** Langlands *et al* considered two crossing probabilities, $\pi_h$ and $\pi_{hv}$, in their extensive numerical investigations of critical percolation in two dimensions. Cardy was able to find the exact form of $\pi_h$ by treating it as a correlation function of boundary operators in the $Q \to 1$ limit of the $Q$-state Potts model. We extend his results to find an analogous formula for $\pi_{hv}$ which compares very well with the numerical results.

## 1. Introduction

Critical percolation in two dimensions has been intensively studied numerically by Langlands *et al* [1, 2]. One of the results they find is that various crossing probabilities, $\pi_h$, $\pi_{hv}$ (to be defined below), are invariant under conformal transformations of the plane. In [3], by combining a standard identification of critical percolation and the 1-state Potts model with boundary conformal field theory techniques, Cardy was able to find the exact form of $\pi_h$. There are three steps needed to extend this to $\pi_{hv}$:

(i) show that $\pi_{hv}$ corresponds to some particular boundary conditions in the 1-state Potts model, and thus is given by a correlation function of conformal field theory boundary operators;
(ii) identify these boundary operators;
(iii) deduce a differential equation for the correlation function, and compare the result with $\pi_{hv}$.

In this letter we complete steps (i) and (iii). Of course, this is far from being a complete derivation of $\pi_{hv}$, but having shown that it is given by a correlation function of boundary operators, then one knows that $\pi_{hv}$ must satisfy one of a discrete set of known differential equations, and examining the simplest of these we find a single function which has suitable asymptotic behaviour. Comparing this function with the numerical data for $\pi_{hv}$ presented in [2], our agreement is as good as that of Cardy's formula for $\pi_h$.

## 2. Critical percolation and boundary conformal field theory

For an extremely good presentation of critical percolation, and the full content of the hypotheses of universality and conformal invariance, we refer the reader to [1]. At its simplest, one can envisage a rectangle, and take as observables all combinations of

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probabilities of crossing from one set of points on the boundary to another disjoint set. The hypothesis of conformal invariance is that such probabilities are invariant under conformal transformations of the plane. For example, the probabilities, $\pi_h$, of crossing from the left side to the right, and $\pi_{hv}$ of being able to cross simultaneously from left to right and from top to bottom, depend only upon the aspect ratio $r$ of the rectangle. It is easy to see that, with $r =$ height/width,

$$\pi_{hv}(r) = \pi_{hv}(1/r) \qquad \pi_{hv}(r)/\pi_h(r) \to 1 \text{ as } r \to 0. \tag{2.1}$$

Langlands *et al* provide numerical values for $\pi_h$ and $\pi_{hv}$ for aspect ratios $0.136 < r < 7.351$.

The property of the crossing probabilities described above, that they are invariant under conformal mappings of the plane, is that of correlation functions of $h = 0$ boundary operators in a $c = 0$ conformal field theory. (For a review of boundary operators in conformal field theory, see e.g. [4, 5].) It was in this way that Cardy was able to find a formula for $\pi_h(r)$. It will be instructive to repeat the relevant points of his derivation here.

The combination $(1 - \pi_h)$ may be identified with the $Q \to 1$ limit of the $Q$-state Potts model partition function with specific boundary conditions, namely free boundary conditions on the vertical sides of the rectangle, and the spins fixed to different values on the horizontal sides. One can identify this partition function as the correlation function of four operators, placed at the corners of the rectangle, which change the boundary conditions.

By considering the 2- and 3-state Potts models, for which the correspondence with $c = 1/2$ and $c = 4/5$ conformal field theories is very well known, Cardy could identify the relevant boundary changing operators as type $(1, 2)$ boundary primary fields. This implied that the correlation function of four such operators should satisfy a second-order differential equation. This equation is very easy to solve, and by taking the appropriate combination of the two independent solutions of this equation, Cardy could fit the data of [1] to a very high degree of accuracy. In fact, this fit was so good that Langlands *et al* decided to adopt Cardy's result as $\pi_h$, rather than their own numerical data, in their later paper on more complicated calculations [2].

## 3. Boundary conditions for $\pi_{hv}$

One can similarly find $\pi_{hv}$ from the $Q \to 1$ limit of the $Q$-state Potts model partition function with certain other boundary conditions which, exactly as for $\pi_h$, are not themselves defined for $Q = 1$. There is a standard argument relating the partition function in the Potts model to a sum over a graphical expansion [6]. In this, one considers all sets of 'clusters' $\xi$, where a cluster is a connected set of bonds on a lattice, and no two clusters intersect. Then the partition function may be written as a sum over clusters, with weights which depend upon the type of cluster, corresponding to the type of boundary conditions one wishes to impose. For our purposes, it will be enough to consider a partition function defined for a square lattice on a rectangle with the sides of the rectangle consecutively numbered 1 to 4, and to consider an expansion of the form

$$Z(a, b, c, d, Q) = \sum_{\xi} p^{B(\xi)}(1-p)^{B-B(\xi)} a^{N_a(\xi)} b^{N_b(\xi)} c^{N_c(\xi)} d^{N-N_a(\xi)-N_b(\xi)-N_c(\xi)} \tag{3.1}$$

where $p$ is the normalized Boltzmann weight for an edge connecting sites with the same spin, and which will be set to the critical probability $p_{\mathrm{c}} = 1/2$ for a bond to be open, when we finally take $Q \to 1$. In (3.1), $B$ is the total number of bonds, $B(\xi)$ is the number of bonds in $\xi$, $N$ is the total number of clusters, $N_a$, $N_b$ and $N_c$ are the numbers of clusters intersecting no sides of the rectangle, exactly 1 side and two adjacent sides respectively, and $a$, $b$, $c$ and $d$ are weights which depend upon the boundary conditions. Taking $a = b = c = d = Q$ one recovers the $Q$-state Potts model partition function for free boundary conditions, and $Z(1, 1, 1, 1, 1) = 1$ in the 1-state Potts model. However, if one can arrange boundary conditions such that, as $Q \to 1$, $a \to 1$, $b \to 1$ and $c \to 1$ and $d \to 0$, then one will recover the probability that no cluster intersects non-adjacent boundaries, i.e. $1 - \pi_h - \pi_v + \pi_{hv}$.

Let us consider the following boundary condition, which we denote by $(AB \ldots C)$: in this boundary condition the spin at each site in the boundary may take its value freely in any of the states $A, B, \ldots, C$. If we assign the conditions $b_i$ to the four sides of the rectangle in turn,

$$
\begin{array}{ll}
b_1 = (A, B, X_{11}, X_{12}, \ldots, X_{1n}) & b_2 = (B, C, X_{21}, X_{22}, \ldots, X_{2n}) \\
b_3 = (C, D, X_{31}, X_{32}, \ldots, X_{3n}) & b_4 = (D, A, X_{41}, X_{42}, \ldots, X_{4n})
\end{array}
\tag{3.2}
$$

where all the spins $\{X_{ij}\}$ and $\{A, B, C, D\}$ are distinct, then we see that the partition function of the $Q$-state Potts model with these boundary conditions will be in the form (3.1) with weights $a = Q$, $b = n + 2$, $c = 1$ and $d = 0$. Then, if we take $n = (Q - 5)/4$ (which is perfectly valid for $Q = 5, 9, \ldots$) and then $Q \to 1$, we shall recover the required weights for our crossing probability, i.e.

$$
\lim_{Q \to 1} Z(Q, (Q + 3)/4, 1, 0, Q) = 1 - \pi_h - \pi_v + \pi_{hv} = \pi_{hv} \tag{3.3}
$$

where we take for granted the universality of critical percolation and set $\pi_h + \pi_v = 1$ as is the case for percolation by sites on a triangular lattice.

Unfortunately, the boundary changing operator which changes the boundary conditions from type $(A, B, X_{11}, X_{12}, \ldots, X_{1n})$ to $(B, C, X_{21}, X_{22}, \ldots, X_{2n})$ does not exist in the 2-state Potts model for any $n$, and has only been identified to date for $n = 0$ in the 3-state Potts model [4, 7], and consequently we cannot deduce a differential equation for $Z$. However, on the basis of this argument, we can certainly expect that $\pi_{hv}$ is expressible as some correlation function of four $h = 0$ conformal boundary primary fields.

## 4. Differential equations for correlation functions

Correlations of conformal primary fields $\phi_h(z)$ can satisfy differential equations, one differential equation for each null state in the highest-weight representations of the Virasoro algebra with highest-weight state $\phi_h(0)\,|0\rangle$, and the general method by which such differential equations are derived is given in [8]. Since the space of null vectors is a highest-weight space, one can restrict attention to the differential equations arising from the vanishing of highest-weight states. For $h = c = 0$ it is easy to compute the highest-weight states with low conformal weight in the Verma module and we give the first four of these in table 1. The four-point functions of four $h = 0$ boundary primary fields will only depend upon the cross ratio $\eta$, of their coordinates, so we shall write

$$
\langle \phi(z_1)\phi(z_2)\phi(z_3)\phi(z_4) \rangle = F(\eta)
$$

where $\eta = ((z_3 - z_4)(z_2 - z_1))/((z_3 - z_1)(z_2 - z_4))$. The differential equations arising from the first three null vectors are equally easy to derive, and are given, along with their solutions, in table 2. The conditions (2.1) translate into

$$
F(\eta) = F(1 - \eta) \qquad \frac{F(\eta)}{\eta^{1/3}{}_2F_1(1/3, 2/3; 4/3; \eta)} \to \frac{3\Gamma(2/3)}{\Gamma(1/3)^2} \quad \text{as } \eta \to 0. \tag{4.1}
$$

**Table 1.** Low-lying highest-weight vectors in the $h = c = 0$ Verma module.

| Level $n$ | Null vector $\mathcal{N}(n)$ |
|---|---|
| 1 | $L_{-1}\,\|0\rangle$ |
| 2 | $(L_{-1}L_{-1} - (2/3)L_{-2})\,\|0\rangle$ |
| 5 | $\left(L_{-1}^3 - 6L_{-2}L_{-1} + 6L_{-3}\right)\mathcal{N}(2)$ |
| 7 | $\left(L_{-1}^5 - (40/3)L_{-2}L_{-1}^3 + (256/9)L_{-2}L_{-2}L_{-1} + (52/3)L_{-3}L_{-1}L_{-1} - (256/9)L_{-3}L_{-2} - (104/3)L_{-4}L_{-1} + (208/9)L_{-5}\right)\mathcal{N}(2)$ |

It is impossible to find a solution of type $F^{(1)}$ or $F^{(2)}$ which satisfies (4.1), but there is a unique solution of type $F^{(5)}$ which does, and which can be presented variously as

$$\begin{aligned}\mathcal{F}(z) &= \frac{\Gamma(2/3)}{\Gamma(1/3)^2}\int_0^z \frac{\mathrm{d}t}{[t(1-t)]^{2/3}} - \frac{2}{3\Gamma(1/3)\Gamma(2/3)}\int_0^z \frac{\mathrm{d}t}{[t(1-t)]^{2/3}}\int_0^t \frac{\mathrm{d}u}{[u(1-u)]^{1/3}} \\ &= \frac{\Gamma(2/3)}{\Gamma(1/3)^2}\int_0^z \frac{\mathrm{d}t}{[t(1-t)]^{2/3}} - \frac{1}{\Gamma(1/3)\Gamma(2/3)}\int_0^z \frac{\mathrm{d}t}{[t(1-t)]^{2/3}} \\ &\quad \times {}_2F_1(1, 4/3; 5/3; t) \\ &= \frac{3\Gamma(2/3)}{\Gamma(1/3)^2} z^{1/3}{}_2F_1(1/3, 2/3; 4/3; z) - \frac{z}{\Gamma(1/3)\Gamma(2/3)}{}_3F_2(1, 1, 4/3; 2, 5/3; z)\end{aligned} \tag{4.2}$$

where ${}_2F_1$ is the standard hypergeometric function, and ${}_3F_2$ is a generalized hypergeometric function.

**Table 2.** Differential operators and solutions.

| Level $n$ | |
|---|---|
| 1 | $\dfrac{\mathrm{d}}{\mathrm{d}z}$ |
| $F^{(1)}$ | $c_1$ |
| 2 | $(z(z-1))^{-2/3}\dfrac{\mathrm{d}}{\mathrm{d}z}(z(z-1))^{2/3}\dfrac{\mathrm{d}}{\mathrm{d}z}$ |
| $F^{(2)}$ | $c_1 + c_2\displaystyle\int_0^z \frac{\mathrm{d}x}{(x(1-x))^{2/3}}$ |
| 5 | $(z(z-1))^{-2}\dfrac{\mathrm{d}^3}{\mathrm{d}z^3}(z(z-1))^{4/3}\dfrac{\mathrm{d}}{\mathrm{d}z}(z(z-1))^{2/3}\dfrac{\mathrm{d}}{\mathrm{d}z}$ |
| $F^{(5)}$ | $c_1 + \displaystyle\int_0^z \frac{\mathrm{d}x}{(x(1-x))^{2/3}}\left[c_2 + \int_0^x \mathrm{d}t\, \frac{c_3 + c_4 t + c_5 t^2}{(t(1-t))^{4/3}}\right]$ |

## 5. Comparison with data and discussion

The case treated in [2] is of the crossing probabilities defined for a rectangle of aspect ratio $r$. If we take the points $z_j$ to be at $(-k^{-1}, -1, 1, k^{-1})$, then we can map these to the corners of a rectangle by

$$w = \int_0^z \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

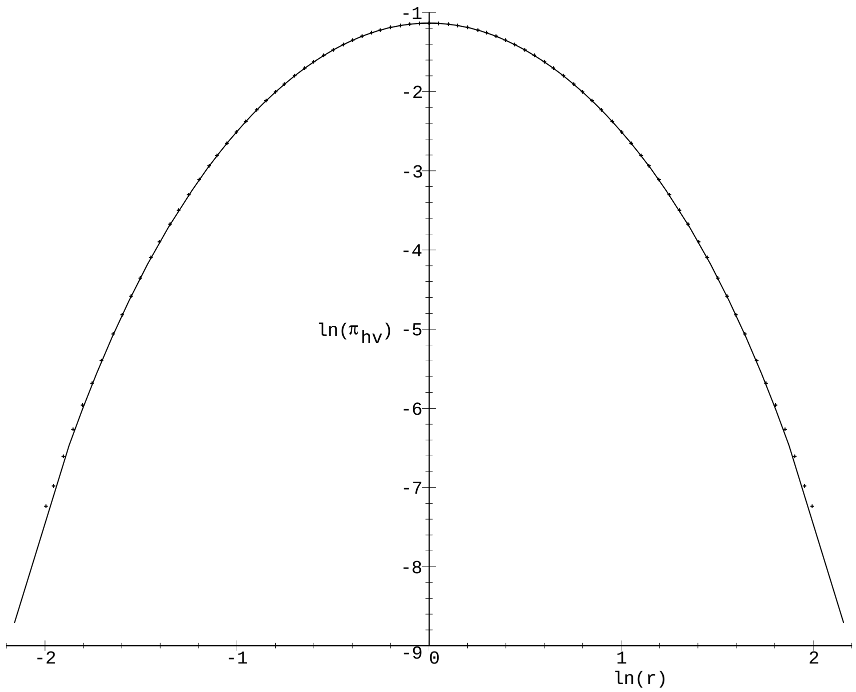


**Figure 1.** A plot of $\ln(\pi_{hv}(r))$ against $\ln(r)$ for the analytic formula and the data.

in which case the aspect ratio of the rectangle is given by $r = K(1-k^2)/2K(k^2)$, where $K(u)$ is the complete elliptic integral of the first kind. The cross ratio of these four points is $\eta = ((1-k)/(1+k))^2$.

In figure 1 we plot $\log(\pi_{hv})$ against $\log(r)$ for both the numerical results obtained in [2] and for the function $\mathcal{F}(x)$ above. The agreement is excellent†.

There are still several points which remain unclear in the results presented here. First, the integral expression of $\pi_h$ found by Cardy can be easily identified as one of the standard solutions of Dotsenko and Fateev [9]. However, despite the fact that $\mathcal{F}$ satisfies the standard differential equations for a type (3, 4) primary field, we have not yet been able to find a way to express (4.2) in Dotsenko–Fateev form. If one allows more general vertex operator constructions, such as for $W$-algebras, then it is easy to see how the troublesome second term in $\mathcal{F}$ might arise as a screened four-point function: one way, using purely bosonic vertex operators, is

$$\lim_{w\to\infty} \int_0^z \mathrm{d}t \int_0^t \mathrm{d}u \, \left\langle 3\alpha + \beta + \gamma \,\middle|\, V_\alpha(w) V_\alpha(1) V_0(z) V_\beta(t) V_\gamma(u) V_\alpha(0) \,\middle|\, 0 \right\rangle$$

where $\alpha^2 = 1/2$, $\alpha\beta = -2/3$, $\alpha\gamma = -1/3$, $\beta\gamma = 0$ and where $V_\lambda(z) \equiv: \exp(\mathrm{i}\lambda\phi(z)) :$ is a bosonic vertex operator, $V_\alpha$ represents a weight 0 field, and $\int V_\beta$ and $\int V_\gamma$ are screening charges. However, we have not yet found any pressing case for such an interpretation for any particular extended algebra.

Second, although $\mathcal{F}(z)$ satisfies the fifth-order differential equation in table 2, it also

† In fact Langlands *et al* only computed $\pi_{hv}$ for $r < 1$, since it is invariant under $r \to 1/r$, and consequently half of the data points we give are duplicated.

satisfies the third-order differential equation

$$\frac{\mathrm{d}}{\mathrm{d}z}(z(z-1))^{1/3}\frac{\mathrm{d}}{\mathrm{d}z}(z(z-1))^{2/3}\frac{\mathrm{d}}{\mathrm{d}z}\mathcal{F}=0. \tag{5.1}$$

This is a much more appealing equation as the three independent crossing probabilities $1, \pi_h, \pi_{hv}$ span the solutions to (5.1), but this equation does not arise from the vanishing of a vector at level 3 in the $h = 0$ Virasoro Verma module. It would be very nice indeed if this equation could be derived from a null vector vanishing for some generalization of the Virasoro algebra, such as those in [10, 11], but we have not been able to do this yet.

## 6. Conclusions

From a simple extension of Cardy's method of [3], we have found an excellent candidate for the crossing probability $\pi_{hv}$ which was investigated numerically by Langlands *et al* in [2]. Unfortunately, we have not been able to find a real derivation of our result, but work is in progress which should elucidate rather more the nature of the conformal field theory underlying this result.

I would very much like to thank Yvan Saint-Aubin for many discussions on percolation and crossing probabilities, and lattice models, conformal field theory and partition functions in general, and J Cardy and G Grimmett for very interesting discussions on this problem, and for their comments on earlier drafts.

I would also like to thank H Kausch for many useful discussions on the structure of $c = 0$ conformal field theory, and the many bizarre possibilities which arise in this case, and W Eholzer for drawing [11] to my attention.

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